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Abstract

We study the square-lattice XY model in the presence of random phase shifts. We consider two different disorder distributions with zero average shift and investigate the low-temperature quasi-long-range order phase which occurs for sufficiently low disorder. By means of Monte Carlo simulations we determine several universal quantities, which are then compared with the analytic predictions of the random spin-wave theory. We observe a very good agreement which indicates that the universal long-distance behaviour in the whole low-disorder low-temperature phase is fully described by the random spin-wave theory.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The two-dimensional XY model with random phase shifts (RPXY) describes the thermodynamic behaviour of several disordered systems, such as magnetic systems with random Dzyaloshinskii–Moriya interactions [1], Josephson junction arrays with geometrical disorder [2, 3], crystal systems on disordered substrates [4] and vortex glasses [5]. See [6, 7] for recent reviews. The RPXY model is defined by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle xy \rangle} \mathrm{Re}\, \psi_x^* U_{xy} \psi_y = -\sum_{\langle xy \rangle} \cos(\theta_x - \theta_y - A_{xy}), \tag{1}$$

where $\psi_x \equiv \mathrm{e}^{\mathrm{i}\theta_x}$, $U_{xy} \equiv \mathrm{e}^{\mathrm{i}A_{xy}}$, and the sum runs over the bonds $\langle xy \rangle$ of a square lattice. The phases, A_{xy}, are uncorrelated quenched random variables with zero average. A Gaussian distribution,

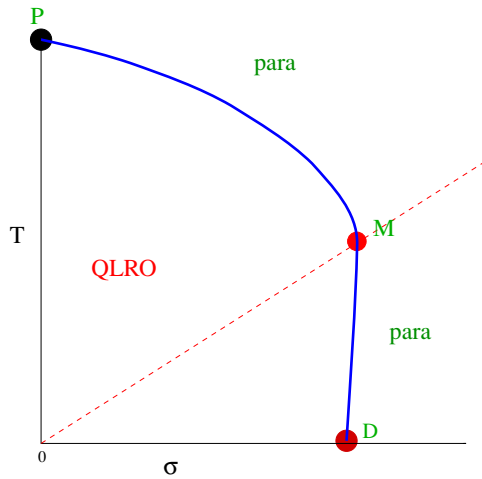$$P(A_{xy}) \propto \exp\left(-\frac{A_{xy}^2}{2\sigma}\right), \tag{2}$$

Figure 1. Generic phase diagram of two-dimensional RPXY models in the temperature–disorder plane. The full line corresponds to the transition line. The dashed line is the N line $T = \sigma$ and refers only to the CRPXY model.

has been considered in most of the studies of the RPXY. We denote the RPXY with Gaussian-distributed phases by GRPXY. The pure XY model is recovered in the limit $\sigma \to 0$, while the so-called gauge glass model [8] with uniformly distributed phase shifts is obtained in the limit $\sigma \to \infty$.

The pure XY model shows a high-temperature paramagnetic phase and a low-temperature phase characterized by quasi-long-range order (QLRO) controlled by a line of Gaussian fixed points. In the latter phase, the spin–spin correlation function, $\langle \bar{\psi}_x \psi_y \rangle$, decays as $1/|x - y|^{\eta(T)}$ for large $|x - y|$, with a T-dependent exponent $\eta(T)$; $\eta(T) \sim T$ for small values of T. The two phases are separated by a Kosterlitz–Thouless (KT) transition [9] at [10] $\beta_{XY} \equiv 1/T_{XY} = 1.1199(1)$. For $\tau \equiv T/T_{XY} - 1 \to 0^+$ the correlation length diverges exponentially as $\ln \xi \sim \tau^{-1/2}$, and the magnetic susceptibility behaves as $\chi \sim \xi^{7/4}$, corresponding to $\eta = 1/4$.

In this paper, we shall discuss the low-temperature behaviour of RPXY models for small disorder. It has already been investigated in several works, most of them focusing on the GRPXY; see, e.g., [1–5, 8, 11–36, 38–46]. The expected T–σ phase diagram, which is sketched in figure 1, presents two phases at finite temperature: a paramagnetic and a QLRO phase. The paramagnetic phase is separated from the QLRO phase by a transition line, which starts from the pure XY point (denoted by P in figure 1) at ($\sigma = 0$, $T = T_{XY} \approx 0.893$) and ends at a zero-temperature disorder-induced transition denoted by D at (σ_0, $T = 0$). Note that QLRO is observed only up to a maximum value, σ_M, of the disorder parameter, which is related to the point $M \equiv (\sigma_M, T_M)$, where the tangent to the transition line is parallel to the T-axis. The transition line from M to D is believed to run (almost) parallel to the T-axis, with $\sigma_0 \lesssim \sigma_M$; see, e.g., [6].[4]

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[4] The first renormalization-group (RG) analyses based on a Coulomb-gas representation of the models [1] predicted reentrant transitions for any value of $\sigma \lesssim \pi/8$, so that $\sigma_0 = 0$. It was then clarified that this was an artefact of the approximations. Indeed, in agreement with the experimental findings [11], numerical works [11–13, 30] and more careful RG analyses [19, 24, 28, 29, 31, 35] showed the absence of reentrant transitions for sufficiently small values of σ, that is predicted $\sigma_0 > 0$. For the CRPXY, Ozeki and Nishimori[19] suggested that the transition line is parallel to the T-axis below the Nishimori point M. It is not clear whether this conjecture is correct. Indeed, the analogous conjecture fails in the case of the 2D $\pm J$ Ising model; see [47] and references therein.

The QLRO phase of the pure XY model is expected to survive for sufficiently small values of σ (see, however, [38] for a recent critical discussion of this scenario). It disappears for large disorder, for [1] $\sigma \gtrsim \sigma_M \approx \pi/8$. The RPXY model for large disorder, and in particular the gauge-glass model ($\sigma \to \infty$), has been much investigated [5, 8, 14–18, 20, 22, 23, 26, 27, 32–34, 36, 37, 39, 40, 42–46]. No long-range glassy order can exist at finite temperature [22, 23]. Some numerical works support a zero-temperature transition; see, e.g., [42, 43, 45].

Beside the GRPXY, we also consider a RPXY model with a slightly different distribution, given by

$$P(A_{xy}) \propto \exp\left(\frac{\cos A_{xy}}{\sigma}\right). \tag{3}$$

We denote the RPXY model with the distribution (3) by CRPXY. The CRPXY is interesting because it allows us to obtain exact results along the so-called Nishimori (N) line [19],

$$T = \sigma, \tag{4}$$

exploiting gauge invariance [19, 48, 49]. For example, the energy density is exactly known along the N line: $E = -I_1(\beta)/I_0(\beta)$, where $\beta \equiv 1/T$ and $I_n(x)$ are the modified Bessel functions. Another important feature of the CRPXY is that along the N line the spin–spin and overlap correlation functions are equal:

$$[\langle \psi_x^* \psi_y \rangle] = [|\langle \psi_x^* \psi_y \rangle|^2], \tag{5}$$

where the angular and square brackets indicate the thermal average and the quenched average over disorder, respectively. The phase diagram is expected to be analogous to that of the GRPXY model. As already noted in [49], the N line must play an important role in the phase diagram, because it is expected to mark the crossover between the region dominated by magnetic correlations and that dominated by randomness. In [19], it was proven that the critical value, σ_M, of σ along the N line is an upper bound for the values of σ where magnetic QLRO can exist (note that this does not exclude the existence of a glassy QLRO for $\sigma > \sigma_M$). Therefore, in figure 1 we can identify it as the point M where $\mathrm{d}\sigma_c/\mathrm{d}T = 0$; here σ_c is the critical value of σ at a given value of T. The critical value σ_0 at $T = 0$ must satisfy $\sigma_0 \leqslant \sigma_M$, leaving open the possibility of reentrant transitions.

In this paper, we focus on the QLRO phase of RPXY models, and investigate its nature. We present numerical Monte Carlo (MC) simulations of the GRPXY and CRPXY models. As we shall see, their results confirm the existence of a QLRO phase in the low-temperature region for small disorder. Moreover, they show that the universal long-distance behaviour in the QLRO phase is described by the random spin-wave theory, in which the long-distance behaviour is essentially identical to that in the model obtained by replacing [1]

$$\cos(\theta_x - \theta_y - A_{xy}) \longrightarrow 1 - \tfrac{1}{2}(\theta_x - \theta_y - A_{xy})^2 \tag{6}$$

in the Hamiltonian (1). Some numerical evidence of QLRO in the GRPXY model was already presented in [30].

The paper is organized as follows. In section 2, we summarize the main predictions of the spin-wave theory, which are then compared with the numerical data in section 3. Our conclusions are presented in section 4. The definitions of the quantities we consider are reported in appendix A. In appendix B, we report the spin-wave calculation of η_s in the CRPXY model for $T \to 0$ in the low-disorder limit.

2. The random spin-wave theory

In the spin-wave limit, the partition function is given by

$$Z(\{A\}) = \int [\mathrm{d}\phi]\, \mathrm{e}^{-H_{sw}/T}, \qquad H_{sw} = \frac{1}{2} \sum_{\langle ij \rangle} (\phi_i - \phi_j - A_{ij})^2, \tag{7}$$

where the link variables, A_{ij}, are uncorrelated quenched random variables with Gaussian probability $P(A_{ij})$. For the GRPXY, the spin and the overlap correlation functions are given by [1, 49]

$$G_s(x - y) = [\langle \mathrm{e}^{\mathrm{i}(\phi_x - \phi_y)} \rangle] = \exp[(T + \sigma)G(x - y)], \tag{8}$$

$$G_o(x - y) = [|\langle \mathrm{e}^{\mathrm{i}(\phi_x - \phi_y)} \rangle|^2] = \exp[2T G(x - y)], \tag{9}$$

where $G(x - y)$ is the (infrared-regularized) two-point function of the Gaussian model without disorder:

$$G(r) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}p\cdot r} - 1}{p^2}. \tag{10}$$

For the CRPXY, one should take into account the nontrivial dependence of $P(A_{ij})$ on A_{ij}. For the overlap correlation function one still obtains (9): for any probability distribution $G_o(x - y)$ does not depend on randomness in the spin-wave approximation. For the spin correlation function, the σ dependence at $T = 0$ is more complex. For $\sigma \to 0$ we obtain

$$G_s(x - y) = \exp[(T + \sigma + \sigma^2/2)G(x - y)], \tag{11}$$

disregarding terms of order σ^3. The derivation is reported in appendix B.

The above-reported results allow us to evaluate the exponents η_s and η_o which are related to the large-distance behaviour of the spin and overlap correlation functions:

$$G_s(r) \sim r^{-\eta_s}, \qquad \eta_s = \begin{cases} \dfrac{T + \sigma}{2\pi} & \text{(GRPXY)}, \\[2mm] \dfrac{T + \sigma + \sigma^2/2}{2\pi} & \text{(CRPXY)}, \end{cases} \tag{12}$$

$$G_o(r) \sim r^{-\eta_o}, \qquad \eta_o = \frac{T}{\pi}. \tag{13}$$

We can thus write for both models, at this level of approximation,

$$G_s(x - y) = \exp[2\pi \eta_s G(x - y)], \qquad G_o(x - y) = \exp[2\pi \eta_o G(x - y)]. \tag{14}$$

Note that the disorder dependence is completely included in the exponents η_s and η_o. Using these expressions we can compute the universal functions, $R_s(\eta_s)$ and $R_o(\eta_o)$, which express the ratios $R_s \equiv \xi_s/L$ and $R_o \equiv \xi_o/L$ in terms of the corresponding exponents η_s and η_o. It is clear that $R_s(\eta_s)$ and $R_o(\eta_o)$ are identical $[R_s(x) = R_o(x)]$ and disorder independent; hence they coincide with those relevant for the pure XY model. These functions are shown in figure 2. Below we show numerically that these predictions are satisfied by our numerical data. This provides clear evidence for the spin-wave nature of the QLRO in the low-temperature and low-disorder region of the RPXY models with Gaussian-like distributions of disorder. A similar strategy was applied in [50] to clarify the nature of the low-temperature phase of fully frustrated XY models.
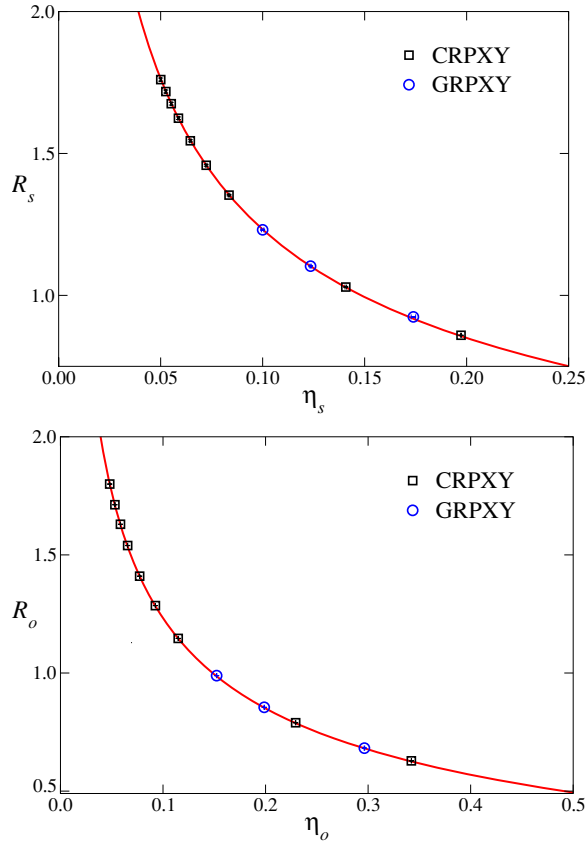
Figure 2. $R_s \equiv \xi_s/L$ versus η_s (top) and $R_o \equiv \xi_o/L$ versus η_o (bottom). The full lines are the results of the random spin-wave theory, obtained by using expressions (14). The MC data are obtained for the CRPXY and GRPXY models with $\sigma = 0.1521$.

3. Monte Carlo results

In this section, we numerically investigate the nature of the QLRO phase that occurs for sufficiently small disorder. In particular, we want to provide a stringent check of the random spin-wave scenario. The quantities which we compute are defined in appendix A.

3.1. Numerical details

We performed MC simulations of the GRPXY and CRPXY models, considering square lattices of linear size L with periodic boundary conditions. We set in both cases $\sigma = 0.1521$, which is well below the maximum value, σ_M ($\sigma_M \gtrsim 0.30$ in the CPRXY model [51] and $\sigma_M \simeq \pi/8$ in the GRPXY model), and considered several values of T below the critical temperature $T_c(\sigma)$, which marks the end of the QLRO phase. MC simulations in the high-temperature phase [51] indicate $T_c = 0.771(2)$ for the GRPXY and $T_c = 0.763(1)$ for the CRPXY.

In the simulations we used a mixture of standard Metropolis and overrelaxed microcanonical updates: a single MC step consisted of five microcanonical sweeps (in which the spins are sequentially updated leaving the energy unchanged) followed by one standard
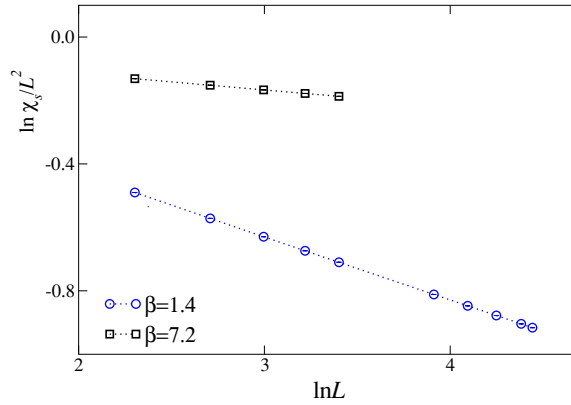
Figure 3. MC estimates of $\ln \chi_s / L^2$ versus $\ln L$ for the CRPXY model at $\sigma = 0.1521$ and two values of $\beta \equiv 1/T$, corresponding to $T \approx 0.139$ and $T \approx 0.714$.

Metropolis sweep. In the MC simulations of the CRPXY model we also used the parallel-tempering method [52, 53]. It allowed us to perform efficient simulations in the region $T \lesssim \sigma$. In the parallel-tempering simulations, we considered N_T systems at the same value of σ and at N_T different inverse temperatures $\beta_{\min} \equiv \beta_1, \ldots, \beta_{\max}$. The largest value, $\beta_{\max}$, corresponded to the minimum temperature value we were interested in. The value, $\beta_{\min}$, was chosen in the paramagnetic phase and was such that thermalization at $\beta = \beta_{\min}$ was sufficiently fast. The intermediate values β_i were chosen such that the acceptance probability of the temperature exchange was at least 5%. Moreover, we always included the value $\beta = 1/\sigma$, which lies along the N line. This provided a check of the numerical programs, since the MC results could be compared with the known exact results [49]. The overlap correlation functions and corresponding χ_o and ξ_o were obtained by simulating two independent replicas for each disorder sample.

In the case of the GRPXY model, we performed standard MC simulations at $T = 2/3, 1/2, 2/5$, for lattice sizes $10 \lesssim L \lesssim 40$. Typically, we considered 50 000 disorder realizations and performed $O(10^5)$ MC steps for each of them. In the case of the CRPXY model, the parallel-tempering method allowed us to investigate the temperature range $T < T_c$ down to $T \approx 0.139$, which is below the N line, i.e. satisfies $T < \sigma = 0.1521$. We performed simulations for lattice sizes $10 \lesssim L \lesssim 30$. Typically, we considered 25 000 disorder realizations. For $\beta \equiv 1/T = 7.2$ we also performed standard MC runs up to $L = 85$.

3.2. QLRO in RPXY models

We estimate the exponents η_s and η_o by studying the finite-size behaviour of the susceptibilities χ_s and χ_o defined in Appendix A. Indeed, for $L \to \infty$ they behave as

$$\chi_{s,o} \sim L^{2-\eta_{s,o}}. \tag{15}$$

Estimates of χ_s are shown in figure 3. On a logarithmic scale, the data fall on a straight line, indicating that the asymptotic behaviour (15) already holds for the values of L we consider. In figure 4, we show the estimates of η_s and η_o for the GRPXY and CRPXY models. For $T \lesssim 0.2$, they agree with the spin-wave approximations (12) and (13). Moreover, they appear to be mostly independent of the model, in agreement with the random spin-wave predictions (note that $\sigma^2/2 = 0.0116$ for $\sigma = 0.1521$). For a more quantitative check, in figure 5 we plot the difference $2\eta_s - \eta_o$ versus T, and compare it with the low-order approximations:
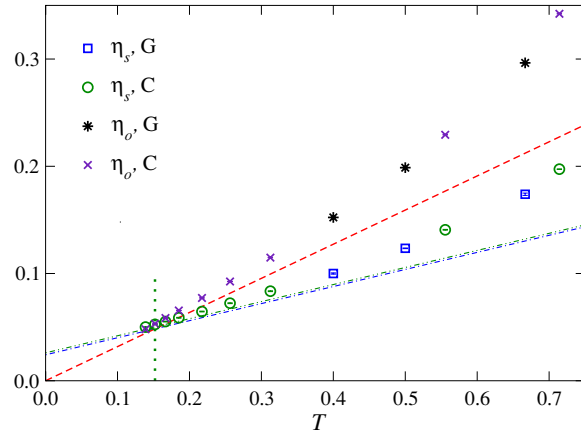
Figure 4. The MC estimates of η_s and η_o versus T for the GRPXY and CRPXY models at $\sigma = 0.1521$. The lines shows the spin-wave approximations (12) and (13). The two lines that give η_s for the GRPXY and CRPXY models cannot easily be distinguished on the scale of the plot. The dotted vertical line corresponds to $T = \sigma = 0.1521$; in the CRPXY model this point belongs to the N line.
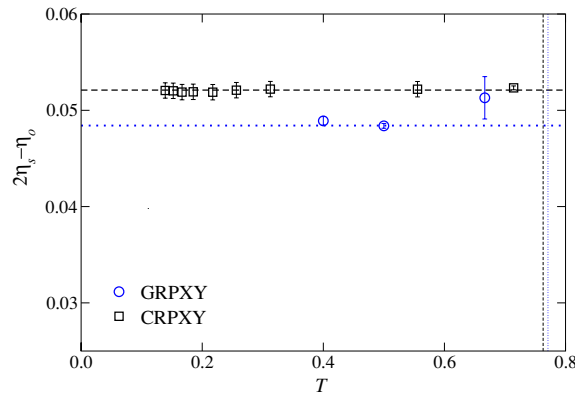

Figure 5. We plot the difference $2\eta_s - \eta_o$ versus T at $\sigma = 0.1521$, and compare it with the low-order spin-wave approximations for the GRPXY and CRPXY models, cf (16) and (17), respectively dotted and dashed lines. The vertical dotted and dashed lines indicate the critical temperatures of the two models, i.e. $T_c = 0.771(2)$ and $T_c = 0.763(1)$ for the GRPXY and CRPXY, respectively.

$$2\eta_s - \eta_o = \frac{\sigma}{\pi} \qquad \text{(GRPXY)}, \qquad (16)$$

$$2\eta_s - \eta_o = \frac{\sigma + \sigma^2/2}{\pi} \qquad \text{(CRPXY)}. \qquad (17)$$

The agreement is very good. Moreover, the above-reported relations appear to hold up to temperatures close to the KT transition, T_c (for $\sigma = 0.1521$ we have $T_c = 0.771(2)$ for the GRPXY model and $T_c = 0.763(1)$ for the CRPXY model), suggesting that they may also hold at the KT transition. Since MC simulations in the high-temperature phase [51] show clear evidence that $\eta_s = 1/4$; this may suggest that at the KT transition
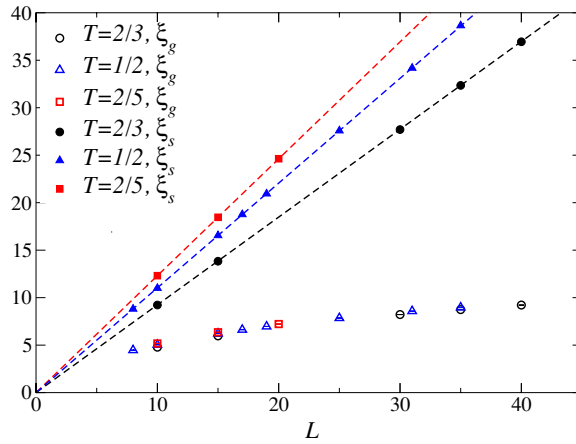
Figure 6. The correlation lengths ξ_s, defined from the standard spin–spin correlation function and ξ_g defined from a gauge invariant spin–spin correlation function, versus L. Results for the GRPXY model with $\sigma = 0.1521$.

$$\eta_o \approx \frac{1}{2} - \frac{\sigma}{\pi} \qquad \text{(GRPXY)}, \tag{18}$$

$$\eta_o \approx \frac{1}{2} - \frac{\sigma + \sigma^2/2}{\pi} \qquad \text{(CRPXY)}. \tag{19}$$

The most important check of the spin-wave nature of the QLRO is provided by the MC data shown in figures 2, where we plot R_s versus η_s and R_o versus η_o: they agree with high accuracy with the curves $R_s(\eta_s)$ and $R_o(\eta_o)$ computed in the spin-wave limit. We believe that these results provide conclusive evidence that the QLRO phase is determined by random spin-wave theory.

Finally, we report some results for the *gauge-invariant* correlation function (A.4); see appendix A. In the spin-wave limit, one finds [49]

$$[\langle e^{i(\phi_x - A_{x,x'} - \dots A_{y',y} - \phi_y)} \rangle] = \exp[(T - \sigma)G(x - y) - |x - y|\sigma/2], \tag{20}$$

which predicts that gauge-invariant spin–spin correlation functions are not critical. For instance, in the large-L limit the correlation length $\xi_g^{(\text{gap})}$ defined from the large-distance exponential decay of the gauge-invariant correlation function is finite and given by $\xi_g^{(\text{gap})} = 2/\sigma$, independently of T. These predictions are confirmed by our MC simulations. We compute the second-moment correlation function ξ_g defined in (A.5). The results are reported in figure 6. It is evident that ξ_g is finite in the large-L limit, satisfies $\xi_g \lesssim 2/\sigma \simeq 13$, and is independent of T. Again, this result shows that the critical behaviour is correctly described by the spin-wave theory.

4. Conclusions

In this paper, we have studied the low-temperature low-disorder phase in RPXY models. We have considered two different disorder distributions and for each of them we have computed numerically the exponents η_s and η_o, and the correlation lengths ξ_s and ξ_o. These results have been compared with the predictions of the random spin-wave theory. Our main results are the following:

(1) The ratios ξ_s/L and ξ_o/L, when expressed in terms of the corresponding exponents η_s and η_o, are in perfect agreement with the analytic predictions. This indicates that expressions (14) hold quite precisely in the whole low-temperature QLRO phase.

(2) Expressions (12) and (13) hold only for very low values of *T*. However, the difference, $2\eta_s - \eta_o$, is apparently well described by spin-wave theory up to the critical transition which marks the end of the paramagnetic phase.

(3) In agreement with the random spin-wave theory, the gauge-invariant spin correlation function (A.4) is not critical.

Finally, note that our calculations refer to probability distributions for which $[A] = 0$. Very little changes if we consider a nonzero average, for instance, one might use the distribution

$$P(A_{xy}) \propto \exp[-(A_{xy} - a)^2/2\sigma]. \qquad (21)$$

In this case we have $[A] = a$. The new model can be mapped into the original one by performing the gauge transformation:

$$\psi'_{(x_1, x_2)} = \mathrm{e}^{-ia(x_1 + x_2)}\psi_{(x_1, x_2)}, \qquad A'_{xy} = A_{xy} - a. \qquad (22)$$

Hence this model has the same phase diagram as the original one. The transformation (22) leaves the overlap correlation functions unchanged, since they are gauge invariant. The behaviour of the spin–spin magnetic correlation functions is more subtle. If $b = (a, a)$, in Fourier space we have

$$\widetilde{G}_s(q; a) = \widetilde{G}_s(q + b; a = 0), \qquad (23)$$

where $\widetilde{G}_s(q; a)$ is the Fourier transform of the spin magnetic correlation function in the theory with a nonvanishing average *a*. In the standard theory ($a = 0$), the critical modes are those with $q = 0$, while for $q \neq 0$ the behaviour is not critical. This implies that the critical modes in the theory with $a \neq 0$ are those associated with a nonvanishing momentum $q = -b$. Hence, in this theory the magnetic susceptibility, which corresponds to $q = 0$, is not critical.

As a final comment, we note that the distribution functions of the phase shifts are not gauge invariant: phase shifts that differ only by a gauge transformation have different probabilities. Another interesting issue is whether the results for the QLRO phase reported here also apply to gauge-invariant distributions.

Appendix A. Notations

In terms of complex site variables $\psi_i \equiv \mathrm{e}^{i\theta_i}$, the RPXY Hamiltonian takes the form

$$\mathcal{H} = -\sum_{\langle ij \rangle} \mathrm{Re}\, \psi_i^* U_{ij} \psi_j, \qquad (A.1)$$

where $U_{ij} \equiv \mathrm{e}^{iA_{ij}}$.

We consider several two-point correlation functions: the magnetic spin–spin correlation function[5],

$$G_s(x - y) \equiv \mathrm{Re}[\langle \psi_x^* \psi_y \rangle] = [\langle \psi_x^* \psi_y \rangle], \qquad (A.2)$$

and the overlap correlation function,

$$G_o(x - y) \equiv [|\langle \psi_x^* \psi_y \rangle|^2], \qquad (A.3)$$

which can be written as $G_o(x - y) = [\langle \bar{q}_x q_y \rangle]$, where $q_x = \psi_x^{(1)*} \psi_x^{(2)}$, and the superscripts refer to two independent replicas with the same disorder. The angular and square brackets

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[5] The last equality in (A.2) can be proved by using the symmetry $\psi_x \to \psi_x^*$ and $U_{xy} \to U_{xy}^*$.

indicate the thermal average and the quenched average over disorder, respectively. We also consider a gauge-invariant spin–spin correlation function,

$$G_g(x - y) \equiv [\mathrm{Re}\langle \psi_x^* U[\Gamma_{x;y}]\psi_y\rangle], \qquad (A.4)$$

where $\Gamma_{x;y}$ is a path that connects sites x and y, and $U[\Gamma_{x;y}]$ is a product of phases associated with the links that belong to $\Gamma_{x;y}$. The paths connecting the points x and y are chosen along the lattice axes, choosing the shortest path (see [54] for details).

We define the corresponding susceptibilities: the magnetic susceptibility $\chi_s \equiv \sum_x G_s(x)$, the overlap susceptibility $\chi_o \equiv \sum_x G_o(x)$ and $\chi_g \equiv \sum_x G_g(x)$. We also define the corresponding second-moment correlation lengths,

$$\xi_\#^2 \equiv \frac{\widetilde{G}_\#(0,0) - \widetilde{G}_\#(q_{\min},0)}{\hat{q}_{\min}^2 \widetilde{G}_\#(q_{\min})}, \qquad (A.5)$$

where $q_{\min} = 2\pi/L$, $\hat{q} \equiv 2\sin q/2$, and $\widetilde{G}_\#(q)$ is the Fourier transform of $G_\#(x)$ and $\#$ indicates s, o, g.

Appendix B. Random spin-wave computation of η_s

In this appendix we wish to derive (11). We follow closely [1]. We first consider the spin–spin correlation function $G_s(r, A_\alpha) = \langle \exp[i(\phi(0) - \phi(r))]\rangle$ for fixed values of the random phases A_α. As in [1] we rewrite it as

$$G_s(r) = \frac{\left\langle \exp\left[i(\phi(0) - \phi(r)) - \beta \int \mathrm{d}^2 r\, A \cdot \nabla\phi\right]\right\rangle_0}{\left\langle \exp\left[-\beta \int \mathrm{d}^2 r\, A \cdot \nabla\phi\right]\right\rangle_0}, \qquad (B.1)$$

where $\langle\cdot\rangle_0$ indicates the average with the Hamiltonian

$$\mathcal{H} = \frac{\beta}{2}\int \mathrm{d}^2 r (\nabla\phi)^2. \qquad (B.2)$$

Repeating the steps discussed in [1] we end up with

$$G_s(r, A) = \exp\left(T G(r) + \frac{1}{2}\int \mathrm{d}^2 s \sum_\alpha A_\alpha(s) M_\alpha(s, r)\right), \qquad (B.3)$$

where $G(r)$ is the Gaussian propagator (10) and

$$M_\alpha(s, r) = 2\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \mathrm{e}^{-iq\cdot s}(\mathrm{e}^{iq\cdot r} - 1)\frac{q_\alpha}{q^2}. \qquad (B.4)$$

Note that $M_\alpha(s, r)$ is imaginary, $M_\alpha(s, r)^* = -M_\alpha(s, r)$, so that

$$|G_s(r, A)|^2 = \mathrm{e}^{2T G(r)}. \qquad (B.5)$$

Thus, irrespective of the phase distribution, the overlap correlation function does not depend on σ. To compute the spin correlation function we must average $G_s(r, A)$ over the distribution of the phases A_α. We consider the general distribution,

$$P(A) \propto \exp\left(-\frac{Q(A^2)}{2\sigma}\right), \qquad (B.6)$$

which satisfies $Q(z) = z$ for $z \to 0$ and is such that, for $\sigma \to 0$, the distribution is peaked around $A = 0$. Thus, to compute the expansion of $G_s(r)$ for small σ, we can expand $Q(A^2)$ in powers of A^2. We assume

$$Q(z) = z + \alpha z^2 + O(z^3), \qquad (B.7)$$

where α is a distribution-dependent coefficient. For the distribution (3) we have $\alpha = -1/12$. To compute the correction of order σ^2 to η_s we rewrite ($[\cdot]_A$ indicates the average over A)

$$S \equiv \left[\exp \left(\frac{1}{2} \int d^2s \sum_\alpha A_\alpha(s) M_\alpha(s,r) \right) \right]_A$$
$$\propto \int [DA] \exp \left(-\frac{1}{2\sigma} \int d^2s [A^2 + \alpha(A^2)^2] + \frac{1}{2} \int d^2s \sum_\alpha A_\alpha M_\alpha \right). \qquad (B.8)$$

We introduce a new field, B_α, defined by

$$A_\alpha = \sqrt{\sigma} B_\alpha + \frac{\sigma}{2} M_\alpha \qquad (B.9)$$

and perform the integral over B. Disregarding terms of order σ^3 we obtain

$$S = \exp \left[\frac{\sigma}{8} (1 - 6\alpha\sigma) \int d^2s \, M(s,r)^2 \right]. \qquad (B.10)$$

Since

$$\int d^2s \, M(s,r)^2 = 8G(r), \qquad (B.11)$$

we obtain finally

$$G_s(r) = e^{(T+\sigma-6\alpha\sigma^2)G(r)}. \qquad (B.12)$$

If we set $\alpha = -1/12$, we obtain result (11).

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